ON UNIQUE CONTINUATION FOR SCHRÖDINGER OPERATORS OF FRACTIONAL AND HIGHER ORDERS

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ABSTRACT. In this note we study the property of unique continuation for solutions of $|(-\Delta)^{\alpha/2}u| \leq |Vu|$, where V is in a function class of potentials including $\bigcup_{p>n/\alpha} L^p(\mathbb{R}^n)$ for $n-1 \leq \alpha < n$. In particular, when $n=2$, our result gives a unique continuation theorem for the fractional $(1<\alpha<2)$ Schrödinger operator $(-\Delta)^{\alpha/2}+V(x)$ in the full range of α values.

1. INTRODUCTION

As it is well-known, analytic functions that are representable by power series have the property of unique continuation. This means that they cannot vanish in any non-empty open set without being identically zero. Note that a solution u of the Cauchy-Riemann operator $\overline{\partial}$ in $\mathbb{R}^2$ (i.e., $\overline{\partial}u=0$) has the property since it is complex analytic. The same result holds for the Laplace operator Δ in $\mathbb{R}^n$ since its solutions are harmonic functions that are still real analytic. So it would be desirable to obtain such property for partial differential operators whose solutions are not necessarily analytic, or even smooth.

Historically, the first such result is due to Carleman [1], who showed the property for the Schrödinger operator $-\Delta+V(x)$ in $\mathbb{R}^2$ if $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$. This was extended to higher dimensions $n \geq 3$ by Müller [11]. Since then, a great deal of work was devoted to the case $V \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $p<\infty$. This is because the potentials V that arise in quantum physics need not be locally bounded and more importantly it can be applied to the problem of absence of positive eigenvalues of the Schrödinger operator. Among others, Jerison and Kenig [3] proved the unique continuation for more general differential inequalities of the form $|\Delta u| \leq |Vu|$ when $V \in L^{n/2}_{\mathrm{loc}}(\mathbb{R}^n)$ if $n>2$, and $V \in \bigcup_{p>1} L^p_{\mathrm{loc}}(\mathbb{R}^2)$ if $n=2$. This result later turns out to be optimal in the context of L^p potentials V ([5, 6]), and was extended to the higher orders $|\Delta^m u| \leq |Vu|$ $(m \in \mathbb{N})$ when $V \in L^{n/2m}_{\mathrm{loc}}(\mathbb{R}^n)$ if $n>2m$ ([7]), and $V \in \bigcup_{p>1} L^p_{\mathrm{loc}}(\mathbb{R}^n)$ if $n=2m$ ([13]).

In this note we are concerned with more general cases of fractional $(0<\alpha<2)$ and higher $(\alpha>2)$ orders:

$$|(-\Delta)^{\alpha/2}u| \leq |Vu|, \tag{1.1}$$

where $(-\Delta)^{\alpha/2}$ is defined by means of the Fourier transform $\mathcal{F}f$ $(=\widehat{f})$:

$$\mathcal{F}[(-\Delta)^{\alpha/2}f](\xi) = |\xi|^\alpha \widehat{f}(\xi).$$

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To the best of our knowledge, all the known results on unique continuation for (1.1) deal only with the case of even integers $\alpha = 2m$, $m \in \mathbb{N}$. When it comes to the other cases of α, the difficulty comes from the fact that $(-\Delta)^{\alpha/2}$ is a nonlocal operator which means that $(-\Delta)^{\alpha/2} f(x)$ depends not just on $f(y)$ for y near x but on $f(y)$ for all y. Moreover, it does not satisfy Leibnitz's rule of differentiation, in general. In order to get around these difficulties, we consider a function class $\mathcal{K}_\alpha$, $0 < \alpha < n$, of potentials V defined by

$$V \in \mathcal{K}_\alpha \quad \Leftrightarrow \quad \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{n-\alpha}} dy = 0. \tag{1.2}$$

The case $\alpha = 2$ is just the usual Kato class introduced by Kato [4] to study the self-adjointness of the Schrödinger operator. Since then, it has played an important role in the study of many other properties of the operator (cf. [14]). In order to extend these studies to $(-\Delta)^{\alpha/2} + V(x)$, the class (1.2) was introduced and used by several authors. (See [16] and the references therein.) By making use of (1.2), we obtain here unique continuation results for (1.1) with $n - 1 \le \alpha < n$.

Before stating our result precisely, it should be emphasized that there are physical interests in the case $1 < \alpha < 2$ as well as the case $\alpha = 2$. Recently, following the path integral approach ([2]) to quantum mechanics, Laskin [8, 9, 10] generalized the Feynman path integral to the Lévy one. This generalization leads to fractional quantum mechanics governed by the fractional Schrödinger equation $i\partial_t \psi = ((-\Delta)^{\alpha/2} + V(x))\psi$, where $1 < \alpha < 2$. The usual quantum mechanics corresponds to the case $\alpha = 2$. In particular, when $n = 2$, Theorem 1.1 below gives a unique continuation result for the fractional Schrödinger operator $(-\Delta)^{\alpha/2} + V(x)$ in the full range of α values.

Theorem 1.1. *Let $n \ge 2$ and $n - 1 \le \alpha < n$. Assume that u is a nonzero solution of* (1.1) *such that*

$$u \in L^1(\mathbb{R}^n) \quad \textit{and} \quad (-\Delta)^{\alpha/2} u \in L^1(\mathbb{R}^n). \tag{1.3}$$

Then it cannot vanish in any non-empty open set of $\mathbb{R}^n$ if V is in the class $\mathcal{K}_\alpha$.

Remark 1.2. From the proof of the theorem, it is not difficult to see that the whole space $\mathbb{R}^n$ can be replaced with an open connected subset.

Note that the class $\mathcal{K}_\alpha$ has the property that $\bigcup_{p>n/\alpha} L^p(\mathbb{R}^n) \subset \mathcal{K}_\alpha \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In fact, if $V \in L^p(\mathbb{R}^n)$, we see that

$$\sup_{x\in\mathbb{R}^n} \int_{|x-y|<r} |x-y|^{-(n-\alpha)} |V(y)| dy \le C \bigg(\int_{|y|<r} |y|^{-(n-\alpha)p'} dy \bigg)^{1/p'} \tag{1.4}$$

by Hölder's inequality. Using polar coordinates, if $p > n/\alpha$, one can see that the right-hand side of (1.4) tends to 0 as $r \to 0$. So it follows that $V \in \mathcal{K}_\alpha$. On the other hand, if $V \in \mathcal{K}_\alpha$, there is $0 < r_0 < 1$ so that the left-hand side of (1.4) is less than 1. Hence we get

$$\sup_{x\in\mathbb{R}^n} \int_{|x-y|<r_0} |V(y)| dy \le 1$$

since $|x|^{-(n-\alpha)} \geq 1$ for $|x| < r_0$. This implies that $V \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Throughout this paper, the letter C stands for positive constants possibly different at each occurrence.

2. Preliminary lemmas

In this section we present some preliminary lemmas which will be used for the proof of Theorem 1.1.

Lemma 2.1. *Let $\phi_\alpha(y) = |y|^{-(n-\alpha)}$ for $0 < \alpha < n$. Then we have for $x \in \mathbb{R}^n$ and $N \geq 1$,*

$$u(x) = C \int_{\mathbb{R}^n} \big[\phi_\alpha(x-y) - \sum_{k=0}^{N-1} \frac{(x\cdot\nabla)^k}{k!}\phi_\alpha(-y)\big](-\Delta)^{\alpha/2}u(y)dy \tag{2.1}$$

if u satisfies (1.3) *and has a compact support in $\mathbb{R}^n \setminus \{0\}$.*

Proof. It is enough to show that (2.1) holds for $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. The remaining follows from this and a standard limiting argument involving a C_0^∞ approximate identity.

First we claim that

$$u(x) = C \int_{\mathbb{R}^n} \phi_\alpha(x-y)(-\Delta)^{\alpha/2}u(y)dy. \tag{2.2}$$

Indeed, using the well-known fact (cf. [15], p.23) that $\widehat{|x|^{-\alpha}}(\xi) = C|\xi|^{-(n-\alpha)}$ in the sense of distributional Fourier transforms, we see that

$$\begin{aligned} u(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\widehat{u}(\xi)d\xi &= \int_{\mathbb{R}^n} |\xi|^{-\alpha}\mathcal{F}\big[(-\Delta)^{\alpha/2}u(\cdot + x)\big](\xi)d\xi \\ &= C\int_{\mathbb{R}^n} |y|^{-(n-\alpha)}(-\Delta)^{\alpha/2}u(y+x)dy, \end{aligned}$$

which gives (2.2).

Now, we note that the $(N-1)^{th}$ degree Taylor polynomial of u at 0 must be zero, since u vanishes near $x = 0$. That is to say,

$$\sum_{|\beta|\leq N-1} \frac{D^\beta u(0)}{\beta!}x^\beta \equiv 0,$$

where β is the usual multiindex notation. By (2.2), this can be rewritten as

$$C\int_{\mathbb{R}^n} \sum_{|\beta|\leq N-1} \frac{D^\beta \phi_\alpha(-y)}{\beta!}x^\beta(-\Delta)^{\alpha/2}u(y)dy \equiv 0. \tag{2.3}$$

Then, we subtract (2.3) from both sides of (2.2) to conclude

$$u(x) = C\int_{\mathbb{R}^n} \big[\phi_\alpha(x-y) - \sum_{|\beta|\leq N-1} \frac{D^\beta\phi_\alpha(-y)}{\beta!}x^\beta\big](-\Delta)^{\alpha/2}u(y)dy$$

which is same as (2.1). ∎

Nextly, we recall from [12] the following estimate on Taylor polynomial approximations to $|x|^{-\beta}$, $\beta > 0$.

Lemma 2.2. *Let $\psi_\beta(x) = |x|^{-\beta}$ for $0 < \beta \leq 1$. Then one has*

$$\Big|\psi_\beta(x-y) - \sum_{k=0}^{N-1} \frac{(x\cdot\nabla)^k}{k!}\psi_\beta(-y)\Big| \leq C\Big(\frac{|x|}{|y|}\Big)^N \psi_\beta(x-y) \tag{2.4}$$

for $x, y \in \mathbb{R}^n$ and $N \geq 1$. Moreover, this estimate is not valid if $\beta > 1$.

3. Proof of Theorem 1.1

Without loss of generality, we need to show that u must be identically zero if u vanishes in a sufficiently small neighborhood of $0 \in \mathbb{R}^n$.

By using (1.1), (2.1) and (2.4) with $\beta = n - \alpha$, if $n - 1 \leq \alpha < n$, we see that

$$|u(x)| \leq C \int_{\mathbb{R}^n} (\frac{|x|}{|y|})^N \phi_\alpha(x-y)|V(y)u(y)|dy. \tag{3.1}$$

Let $f(y) = |y|^{-N}|V(y)u(y)|$. Since u vanishes near the origin, it follows that

$$\|f\|_{L^1} = \int_{\mathbb{R}^n} |y|^{-N}|V(y)u(y)|dy < \infty. \tag{3.2}$$

Here, we also used the fact (see Theorem 2.2 in [16]) that the condition (1.3) implies $Vu \in L^1$ if $V \in \mathcal{K}_\alpha$. Hence, from (3.1) we get

$$\begin{aligned}\int_{|x|<r} |V(x)||x|^{-N}|u(x)|dx &\leq C\int_{\mathbb{R}^n}\bigg(\int_{|x|<r}\phi_\alpha(x-y)|V(x)|dx\bigg)f(y)dy \\ &\leq C\bigg(\sup_{y\in\mathbb{R}^n}\int_{|x|<r}\phi_\alpha(x-y)|V(x)|dx\bigg)\|f\|_{L^1}. \end{aligned}\tag{3.3}$$

Now, we set

$$\eta(r) = \sup_{y\in\mathbb{R}^n}\int_{|x|<r}\phi_\alpha(x-y)|V(x)|dx.$$

Then the condition (1.2) implies $\lim_{r\to 0}\eta(r) = 0$. In fact, we note that

$$\sup_{|y|<2r}\int_{|x|<r}\phi_\alpha(x-y)|V(x)|dx \leq \sup_{y\in\mathbb{R}^n}\int_{|x-y|<4r}\phi_\alpha(x-y)|V(x)|dx$$

and

$$\begin{aligned}\sup_{|y|\geq 2r}\int_{|x|<r}\phi_\alpha(x-y)|V(x)|dx &\leq Cr^{\alpha-n}\int_{|x|<r}|V(x)|dx \\ &\leq Cr^{\alpha-n}r^{n-\alpha}\int_{|x|<r}\frac{|V(x)|}{|x|^{n-\alpha}}dx.\end{aligned}$$

Then it follows from (1.2) that

$$\lim_{r\to 0}\eta(r) \leq C\lim_{r\to 0}\sup_{y\in\mathbb{R}^n}\int_{|x-y|<4r}\frac{|V(x)|}{|x-y|^{n-\alpha}}dx = 0.$$

Hence, if we choose $r_0 > 0$ small enough, we see from (3.3) that

$$\int_{|x|<r_0} |V(x)||x|^{-N}|u(x)|dx \le \frac{1}{2}\|f\|_{L^1}.$$

Combining this with (3.2), we get

$$\int_{|x|<r_0} |V(x)||x|^{-N}|u(x)|dx \le \int_{|y|\ge r_0} |y|^{-N}|V(y)u(y)|dy,$$

so that

$$\int_{|x|<r_0} |V(x)|\big(\frac{r_0}{|x|}\big)^N|u(x)|dx \le \|Vu\|_{L^1} < \infty. \tag{3.4}$$

Here we may assume that $|V| \ge 1$, since $|V|+1$ also satisfies (1.1) and $\lim_{r\to 0}\eta(r) = 0$. Indeed, to show the second one for $|V|+1$, we only need to show

$$\lim_{r\to 0}\sup_{y\in\mathbb{R}^n}\int_{|x|<r} \phi_\alpha(x-y)dx = 0. \tag{3.5}$$

Since $\phi_\alpha(x) = |x|^{-(n-\alpha)}$, it follows that

$$\begin{aligned}\sup_{y\in\mathbb{R}^n}\int_{|x|<r} \phi_\alpha(x-y)dx &\le \sup_{|y|<2r}\int_{|x|<r} |x-y|^{-(n-\alpha)}dx + \sup_{|y|\ge 2r}\int_{|x|<r} r^{-(n-\alpha)}dx \\ &\le \sup_{y\in\mathbb{R}^n}\int_{|x-y|<4r} |x-y|^{-(n-\alpha)}dx + Cr^\alpha \\ &\le Cr^\alpha.\end{aligned}$$

This gives (3.5).

Therefore, from (3.4) we see that

$$\int_{|x|<\frac{r_0}{2}} 2^N|u(x)|dx \le \int_{|x|<r_0} |V(x)|\big(\frac{r_0}{|x|}\big)^N|u(x)|dx < \infty.$$

By letting $N \to \infty$, we get that u vanishes in $\{|x| < \frac{r_0}{2}\}$. Then, using a standard connectedness argument, we can conclude that u must be identically zero in $\mathbb{R}^n$. This completes the proof.

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